

MAXIMALLY NONABELIAN TODA SYSTEMS

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Abstract

A detailed consideration of the maximally nonabelian Toda systems based on the classical semisimple Lie groups is given. The explicit expressions for the general solution of the corresponding equations are obtained.

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1 Introduction

Last two decades the Toda systems permanently attract great attention of the physicists and mathematicians due to their integrability and deep links with a number of problems in the theory of differential equations, differential and algebraic geometry, Lie algebras, Lie groups and their representations, etc.; and relevance to many problems of modern theoretical and mathematical physics. In particular, they arise in a natural way in various approaches of the particle physics (e.g., field theory models and supergravity including black holes and p -branes business) and statistical mechanics (e.g., correlations in the inhomogeneous XY -model at infinite temperature). Whereas there is a lot of papers devoted to classical and quantum behaviour of abelian Toda systems, nonabelian Toda systems remain in many aspects unexplored. It is mainly caused by the absence of nontrivial examples, for which one can write the general solution for the equations under consideration in a rather explicit form. However, there exists a class of nonabelian Toda systems having a very simple structure. We call these systems maximally nonabelian Toda systems by the following reason.

A Toda system is related to some Lie group G whose Lie algebra \mathfrak{g} is equipped with a \mathbb{Z} -gradation, and hence there is defined the subgroup \tilde{H} corresponding to the subalgebra formed by the zero grade elements. The Toda fields parametrise a mapping from \mathbb{R}^2 or \mathbb{C} to \tilde{H} . If the subgroup \tilde{H} is abelian, we deal with an abelian Toda system, otherwise we have a nonabelian Toda system. In the case of the trivial \mathbb{Z} -gradation the subgroup \tilde{H} coincides with G and we come to the Wess-Zumino-Novikov-Witten equations. The maximally nonabelian Toda systems correspond to the case when the subgroup \tilde{H} does not coincide with G and is not a proper subgroup of any subgroup of G generated by the zero grade subspace for some nontrivial \mathbb{Z} -gradation of \mathfrak{g} .

In the present paper we give a detailed consideration of the maximally nonabelian Toda systems associated with the classical semisimple finite dimensional Lie groups.

2 Toda systems and their integration

2.1 \mathbb{Z} -gradations

The starting point for the construction [1, 2, 3, 4] of Toda equations is a complex Lie group whose Lie algebra is endowed with a \mathbb{Z} -gradation. In the first two sections we give some necessary information about \mathbb{Z} -gradations of the complex semisimple Lie algebras, for more details see, for example, [5, 3]. Recall that a Lie algebra \mathfrak{g} is said to be endowed with a \mathbb{Z} -gradation if there is given a representation of \mathfrak{g} as a direct sum

$$\mathfrak{g} = \bigoplus_{a \in \mathbb{Z}} \mathfrak{g}_a,$$

where $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b}$ for all $a, b \in \mathbb{Z}$.

Let G be a complex Lie group, and \mathfrak{g} be its Lie algebra. For a given \mathbb{Z} -gradation of \mathfrak{g} introduce the following subalgebras of \mathfrak{g} :

$$\tilde{\mathfrak{h}} \equiv \mathfrak{g}_0, \quad \tilde{\mathfrak{n}}_- \equiv \bigoplus_{a < 0} \mathfrak{g}_a, \quad \tilde{\mathfrak{n}}_+ \equiv \bigoplus_{a > 0} \mathfrak{g}_a.$$

Here and henceforth we use tildes to have the notations different from those usually used in the case of the so called canonical gradation of complex semisimple Lie algebras.

Denote by \tilde{H} and \tilde{N}_\pm the connected Lie subgroups of G corresponding to the subalgebras $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{n}}_\pm$ respectively. Suppose that \tilde{H} and \tilde{N}_\pm are closed subgroups of G and, moreover,

$$\begin{aligned}\tilde{H} \cap \tilde{N}_\pm &= \{e\}, & \tilde{N}_- \cap \tilde{N}_+ &= \{e\}, \\ \tilde{N}_- \cap \tilde{H}\tilde{N}_+ &= \{e\}, & \tilde{N}_-\tilde{H} \cap \tilde{N}_+ &= \{e\}.\end{aligned}$$

where e is the unit element of G . This is true, in particular, for the finite dimensional reductive Lie groups, see, for example, [6]. The set $\tilde{N}_-\tilde{H}\tilde{N}_+$ is an open subset of G . Suppose that $G = \overline{\tilde{N}_-\tilde{H}\tilde{N}_+}$, where the bar means the topological closure. This is again true for the finite dimensional reductive Lie groups. Thus, in the case under consideration for any element a which belongs to the dense set $\tilde{N}_-\tilde{H}\tilde{N}_+$ one can write the following unique decomposition

$$a = n_- h n_+^{-1}, \quad (2.1)$$

where $n_- \in \tilde{N}_-$, $h \in \tilde{H}$ and $n_+ \in \tilde{N}_+$. Decomposition (2.1) is called the Gauss decomposition. Note that the Gauss decomposition (2.1) is the principal tool used in the group-algebraic integration procedure for Toda equations.

There is a simple classification of possible \mathbb{Z} -gradations for complex semisimple Lie algebras. In this case for any \mathbb{Z} -gradation of a such an algebra \mathfrak{g} , there exists a unique element $q \in \mathfrak{g}$ which has the following property. An element $x \in \mathfrak{g}$ belongs to the subspace \mathfrak{g}_a if and only if $[q, x] = ax$. This can be written as

$$\mathfrak{g}_a = \{x \in \mathfrak{g} \mid [q, x] = ax\}.$$

The element q is called the grading operator. It is clearly semisimple, and

$$\exp(2\pi\sqrt{-1}\operatorname{ad} q) = \operatorname{id}_{\mathfrak{g}}.$$

On the other hand, any semisimple element of \mathfrak{g} which possesses this property defines a \mathbb{Z} -gradation of \mathfrak{g} .

Since the grading operator q is semisimple, one can always point out a Cartan subalgebra of \mathfrak{g} which contains q [7]. Let \mathfrak{h} be such a Cartan subalgebra, and Δ be the root system of \mathfrak{g} with respect to \mathfrak{h} . For any element x of the root subspace \mathfrak{g}^α corresponding to the root $\alpha \in \Delta$, one has $[q, x] = \langle \alpha, q \rangle x$, where $\langle \alpha, q \rangle$ means the action of the element $\alpha \in \mathfrak{g}^*$ on the element $q \in \mathfrak{g}$. Hence, for any root $\alpha \in \Delta$ the number $\langle \alpha, q \rangle$ is an integer. Furthermore, if we choose a base $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of Δ corresponding to the Weyl chamber whose closure contains q , then the integers $s_i \equiv \langle \alpha_i, q \rangle$ are nonnegative. The grading subspace \mathfrak{g}_a , $a \neq 0$, is the direct sum of the root subspaces \mathfrak{g}_α corresponding to the roots $\alpha = \sum_{i=1}^r n_i \alpha_i$ with $\sum_{i=1}^r n_i s_i = a$. The subspace \mathfrak{g}_0 , besides of the root subspaces corresponding to the roots $\alpha = \sum_{i=1}^r n_i \alpha_i$ with $\sum_{i=1}^r n_i s_i = 0$, includes the Cartan subalgebra \mathfrak{h} .

It can be shown that, up to a possible reordering related to the freedom in the renumbering of the elements of Π , the numbers s_i do not depend neither on the choice of a Cartan subalgebra containing q , nor on the choice of a base possessing the above described property. In other words, a \mathbb{Z} -gradation of a semisimple Lie algebra \mathfrak{g} is

described by nonnegative integer labels associated with the vertices of the corresponding Dynkin diagram. Two \mathbb{Z} -gradations are connected by an inner automorphism of \mathfrak{g} if and only if they have the same set of labels. Two \mathbb{Z} -gradations are connected by an ‘external’ automorphism of \mathfrak{g} if and only if the corresponding sets of labels are connected by an automorphism of the Dynkin diagram.

Let now \mathfrak{h} be a Cartan subalgebra of a semisimple Lie algebra \mathfrak{g} , Δ be the root system of \mathfrak{g} with respect to \mathfrak{h} , and $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a base of Δ . Denote by h_i and $x_{\pm i}$, $i = 1, \dots, r$, the corresponding Cartan and Chevalley generators of \mathfrak{g} . For any set of r nonnegative numbers s_i the element

$$q = \sum_{i,j=1}^r (k^{-1})_{ij} s_j h_i, \quad (2.2)$$

where $k = (k_{ij})$ is the Cartan matrix of \mathfrak{g} , is the grading operator of some \mathbb{Z} -gradation of \mathfrak{g} . Here the numbers s_i are the corresponding labels of the Dynkin diagram. The canonical gradation of \mathfrak{g} arises when one chooses all the number s_i equal to 1. Thus, there is a bijective correspondence between the sets of nonnegative integer labels of the Dynkin diagram of a semisimple Lie algebra and the classes of its conjugated \mathbb{Z} -gradations.

If all the labels of the Dynkin diagram of a semisimple Lie algebra \mathfrak{g} are different from zero, then the subgroup \mathfrak{g}_0 coincides with some Cartan subalgebra of \mathfrak{g} . In this case the subgroup \tilde{H} is abelian and we deal with the so called abelian Toda equations. In all other cases the subgroup \tilde{H} is nonabelian and the corresponding Toda equations are called nonabelian. In particular, if all the labels are equal to zero, then there is only one grading subspace $\mathfrak{g}_0 = \mathfrak{g}$. In this case the corresponding Toda equations coincide with the Wess-Zumino-Novikov-Witten equations. In the present paper we consider the case when only one of the labels is different from zero; the corresponding equations are called here maximally nonabelian Toda equations.

Sometimes it is more convenient to consider, instead of a semisimple Lie algebra, some reductive Lie algebra which contains this semisimple Lie algebra. Here we use \mathbb{Z} -gradations defined by the following procedure. Recall that a reductive Lie algebra \mathfrak{g} can be represented as the direct product of the center $Z(\mathfrak{g})$ of \mathfrak{g} and a semisimple subalgebra \mathfrak{g}' of \mathfrak{g} . Choose some \mathbb{Z} -gradation of \mathfrak{g}' and denote the corresponding grading operator by q . It is clear that q is the grading operator of some \mathbb{Z} -gradation of the whole Lie algebra \mathfrak{g} . Moreover, the subspaces \mathfrak{g}_a , $a \neq 0$, coincide with the subspaces \mathfrak{g}'_a , and the subspace \mathfrak{g}_0 is the direct sum of the subspace \mathfrak{g}'_0 and the center $Z(\mathfrak{g})$.

2.2 $\mathfrak{sl}(2, \mathbb{C})$ -subalgebras

It is often useful to consider \mathbb{Z} -gradations of a Lie algebra \mathfrak{g} associated with embeddings of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ into \mathfrak{g} . Recall that the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is a complex simple Lie algebra formed by all traceless 2×2 matrices. This Lie algebra is of rank 1, and the Cartan and Chevalley generators satisfy the following commutation relations

$$[x_+, x_-] = h, \quad (2.3)$$

$$[h, x_-] = -2x_-, \quad [h, x_+] = 2x_+. \quad (2.4)$$

By an embedding of $\mathfrak{sl}(2, \mathbb{C})$ into \mathfrak{g} we mean a nontrivial homeomorphism from $\mathfrak{sl}(2, \mathbb{C})$ into \mathfrak{g} . The images of the elements h and x_{\pm} under the homomorphism, defining the embedding under consideration, are denoted usually by the same letters. The image of the whole Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is called an $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra of \mathfrak{g} . For a given embedding of $\mathfrak{sl}(2, \mathbb{C})$ into \mathfrak{g} , the adjoint representation of \mathfrak{g} defines the representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ in \mathfrak{g} . From the properties of the finite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ it follows the element h of \mathfrak{g} must be semisimple, and the elements x_{\pm} must be nilpotent. Moreover, it is clear that $\exp(2\pi\sqrt{-1}\text{ad } h) = \text{id}_{\mathfrak{g}}$. Therefore, the element h can be used as the grading operator defining some \mathbb{Z} -gradation of \mathfrak{g} . It can be shown that in the case when \mathfrak{g} is semisimple, the labels of the corresponding Dynkin diagram can be equal only to 0, 1, and 2. In particular, if the labels are equal only to 0 or 2, we deal with the so called integral embedding. In this case it is natural to consider the \mathbb{Z} -gradation defined by the grading operator $q = h/2$.

2.3 Toda equations

Let M be either a real two dimensional manifold, or a complex one dimensional manifold. Choose some local coordinates z^- and z^+ on M . In the complex case we assume that $z^+ = \overline{z^-}$. Denote the partial derivatives over z^+ and z^- by ∂_+ and ∂_- respectively. Consider a \mathbb{Z} -graded complex semisimple Lie algebra \mathfrak{g} . Let l be a positive integer, such that the grading subspaces \mathfrak{g}_a for $-l < a < 0$ and $0 < a < l$ are trivial, and c_- and c_+ be some fixed elements of the subspaces \mathfrak{g}_{-l} and \mathfrak{g}_{+l} respectively. Restrict ourselves to the case when G is a matrix Lie group. In this case the Toda equations are the matrix partial differential equations of the form

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma], \quad (2.5)$$

where γ is a mapping from M to \tilde{H} .

Let h_{\pm} be some elements of \tilde{H} , and the mapping γ satisfies the Toda equations. It is easy to get convinced that the mapping

$$\gamma' = h_+^{-1}\gamma h_-$$

satisfies the Toda equations (2.5) with the elements c_{\pm} replaced by the elements

$$c'_{\pm} = h_{\pm}^{-1}c_{\pm}h_{\pm}. \quad (2.6)$$

In this sense, the Toda equations determined by the elements c_{\pm} and c'_{\pm} which are connected by the above relation, are equivalent.

Denote by \tilde{H}_{\pm} the subgroups of \tilde{H} defined by

$$\tilde{H}_{\pm} = \{h \in \tilde{H} \mid hc_{\pm}h^{-1} = c_{\pm}\}. \quad (2.7)$$

The Toda equations are invariant with respect to the transformations

$$\gamma' = \xi_+^{-1}\gamma\xi_- \quad (2.8)$$

where ξ_{\pm} are arbitrary mappings from M to the subgroups \tilde{H}_{\pm} , satisfying the relations $\partial_{\mp}\xi_{\pm} = 0$.

2.4 Integration scheme

To obtain the general solution of Toda equations one can use the following procedure [1, 2, 3, 4]. Choose some mappings γ_{\pm} from M to \tilde{H} such that

$$\partial_{\mp}\gamma_{\pm} = 0. \quad (2.9)$$

Integrate the equations

$$\mu_{\pm}^{-1}\partial_{\pm}\mu_{\pm} = \gamma_{\pm}x_{\pm}\gamma_{\pm}^{-1}, \quad \partial_{\mp}\mu_{\pm} = 0. \quad (2.10)$$

The solutions of the above equations are fixed by the conditions

$$\mu_{\pm}(p) = a_{\pm}, \quad (2.11)$$

where p is some fixed point of M , and a_{\pm} are some elements of G . The mappings μ_{\pm} satisfying equations (2.10) and conditions (2.11) are unique and take values in the sets $a_{\pm}\tilde{N}_{\pm}$. The Gauss decomposition (2.1) induces the corresponding decomposition of mappings from M to G . In particular, one obtains

$$\mu_{+}^{-1}\mu_{-} = \nu_{-}\eta\nu_{+}^{-1}, \quad (2.12)$$

where the mapping η takes values in \tilde{H} , and the mappings ν_{\pm} take values in \tilde{N}_{\pm} . It can be shown that the mapping

$$\gamma = \gamma_{+}^{-1}\eta\gamma_{-} \quad (2.13)$$

satisfies the Toda equations, and any solution to this equation can be obtained by the described procedure. Note that almost all solutions of the Toda equations can be obtained using the mappings μ_{\pm} submitted to relation (2.11) with $a_{\pm} \in \tilde{N}_{\pm}$, or, in other words, using the mappings μ_{\pm} taking values in the subgroups \tilde{N}_{\pm} [3, 4].

3 Complex general linear group

3.1 Equations

We begin the consideration of maximally nonabelian Toda systems with the case of the Lie group $\mathrm{GL}(r+1, \mathbb{C})$. Recall that the corresponding Lie algebra $\mathfrak{gl}(r+1, \mathbb{C})$ is reductive and can be represented as the direct product of the simple Lie algebra $\mathfrak{sl}(r+1, \mathbb{C})$ and a one dimensional Lie algebra isomorphic to $\mathfrak{gl}(1, \mathbb{C})$ and composed by the $(r+1) \times (r+1)$ complex matrices which are multiplies of the unit matrix.

The Lie algebra $\mathfrak{sl}(r+1, \mathbb{C})$ is of type A_r . Let d be a fixed integer such that $1 \leq d \leq r$. Consider the \mathbb{Z} -gradation of $\mathfrak{sl}(r+1, \mathbb{C})$ arising when we choose the labels of the corresponding Dynkin diagram equal to zero except the label s_d which is chosen equal to 1. Construct the grading operator associated with this gradation using relation (2.2) and the well known explicit expression for the inverse of the Cartan matrix, see e.g., [8, 3]. From relation (2.2) it follows that the grading operator in the case under consideration has the form

$$q = \frac{1}{r+1} \left[(r+1-d) \sum_{i=1}^{d-1} ih_i + d \sum_{i=d}^r (r+1-i)h_i \right].$$

It is convenient to take as a Cartan subalgebra for $\mathfrak{sl}(r+1, \mathbb{C})$ the subalgebra consisting of diagonal $(r+1) \times (r+1)$ matrices with zero trace. Here the standard choice of the Cartan generators is

$$h_i = e_{i,i} - e_{i+1,i+1},$$

where the matrices $e_{i,j}$ are defined by

$$(e_{i,j})_{kl} \equiv \delta_{ik} \delta_{jl}. \quad (3.1)$$

With such a choice of Cartan generators we obtain

$$q = \frac{1}{r+1} \left[(r+1-d) \sum_{i=1}^d e_{i,i} - d \sum_{i=d+1}^{r+1} e_{i,i} \right].$$

Thus, the grading operator has the following block matrix form:

$$q = \begin{pmatrix} \frac{m_2}{m_1+m_2} I_{m_1} & 0 \\ 0 & \frac{-m_1}{m_1+m_2} I_{m_2} \end{pmatrix}, \quad (3.2)$$

where $m_1 = d$ and $m_2 = r+1-d$, so that $m_1 + m_2 = r+1$. Here and henceforth I_m denotes the unit $m \times m$ matrix. We will use this grading operator to define a \mathbb{Z} -gradation of the Lie algebra $\mathfrak{gl}(r+1, \mathbb{C})$. It is not difficult to show that in this case we have three grading subspaces, \mathfrak{g}_0 and $\mathfrak{g}_{\pm 1}$. To describe these subspaces, it is convenient to consider $(r+1) \times (r+1)$ matrices as 2×2 block matrices $x = (x_{ij})$, where x_{ij} are $m_i \times m_j$ matrices. Then the subspace $\mathfrak{g}_0 = \tilde{\mathfrak{h}}$ consists of all block diagonal matrices, and the subspaces $\mathfrak{g}_{-1} = \tilde{\mathfrak{n}}_-$ and $\mathfrak{g}_{+1} = \tilde{\mathfrak{n}}_+$ are formed by all block strictly lower and upper triangular matrices respectively. In other words, one can say that the subspace \mathfrak{g}_a consists of the matrices $x = (x_{ij})$ where only the blocks x_{ij} with $j-i = a$ are different from zero.

It is easy to describe the corresponding subgroups of $\mathrm{GL}(r+1, \mathbb{C})$. The subgroup \tilde{H} is formed by all block diagonal nondegenerate matrices, and the subgroups \tilde{N}_- and \tilde{N}_+ consist respectively of all block upper and lower triangular matrices with unit matrices on the diagonal.

Proceed now to the consideration of the corresponding Toda equations. The general form of the elements c_{\pm} is

$$c_- = \begin{pmatrix} 0 & 0 \\ C_- & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_+ \\ 0 & 0 \end{pmatrix}. \quad (3.3)$$

Parametrise the mapping γ as

$$\gamma = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

where the mappings β_1 and β_2 take values in the groups $\mathrm{GL}(m_1, \mathbb{C})$ and $\mathrm{GL}(m_2, \mathbb{C})$ respectively. Using now the relation

$$\gamma^{-1} c_+ \gamma = \begin{pmatrix} 0 & \beta_1^{-1} C_+ \beta_2 \\ 0 & 0 \end{pmatrix},$$

we can write the Toda equations in the form

$$\partial_+(\beta_1^{-1} \partial_- \beta_1) = -\beta_1^{-1} C_+ \beta_2 C_-, \quad (3.4)$$

$$\partial_+(\beta_2^{-1} \partial_- \beta_2) = C_- \beta_1^{-1} C_+ \beta_2. \quad (3.5)$$

Recall that the Toda equations defined by the elements c_{\pm} and c'_{\pm} connected by relation (2.6) are equivalent. For the case under consideration this statement is equivalent to saying that equations (3.4), (3.5) are determined by fixing the ranks of the matrices C_- and C_+ .

Let us try to associate the \mathbb{Z} -gradation of $\mathfrak{gl}(r+1, \mathbb{C})$ which is considered in this section with some embedding of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{gl}(r+1, \mathbb{C})$. It is clear that the Cartan generator h of the corresponding $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra should coincide with $2q$. The Chevalley generators x_- and x_+ should belong to the subspaces \mathfrak{g}_{-1} and \mathfrak{g}_{+1} respectively. Therefore, we can write for x_- and x_+ the block matrix representation

$$x_- = \begin{pmatrix} 0 & 0 \\ X_- & 0 \end{pmatrix}, \quad x_+ = \begin{pmatrix} 0 & X_+ \\ 0 & 0 \end{pmatrix}. \quad (3.6)$$

To satisfy relation (2.3) we should have

$$X_+X_- = \frac{2m_2}{m_1 + m_2}I_{m_1}, \quad X_-X_+ = \frac{2m_1}{m_1 + m_2}I_{m_2}.$$

From this equalities it follows that the rank $r(X_+X_-)$ is equal to m_1 and the rank $r(X_-X_+)$ is equal to m_2 . On the other hand,

$$r(X_+X_-) \leq \min(r(X_+), r(X_-)) \leq \min(m_1, m_2).$$

Hence, one has $m_1 \leq \min(m_1, m_2)$ and $m_2 \leq \min(m_1, m_2)$. This is possible only if $m_1 = m_2 = m$, and in this case $r(X_+) = m$ and $r(X_-) = m$. It can be easily shown that the equality $m_1 = m_2$ is a sufficient condition for the existence of an $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra in question.

If the condition $m_1 = m_2$ is satisfied, then without any loss of generality, we can choose

$$h = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}, \quad x_- = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}, \quad x_+ = \begin{pmatrix} 0 & I_m \\ 0 & 0 \end{pmatrix}. \quad (3.7)$$

Now, with $c_{\pm} = x_{\pm}$, one comes to the Toda equations of the form

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}\beta_2, \quad (3.8)$$

$$\partial_+(\beta_2^{-1}\partial_-\beta_2) = \beta_1^{-1}\beta_2, \quad (3.9)$$

where the mappings β_1 and β_2 take values in the Lie group $GL(m, \mathbb{C})$. In this case the subgroups \tilde{H}_{\pm} defined by (2.7) are isomorphic to the Lie group $GL(m, \mathbb{C})$ and are composed of the block matrices of the form

$$h = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{11} \end{pmatrix}. \quad (3.10)$$

Recall that these subgroups determine the symmetry transformations (2.8) of the Toda equations.

3.2 General solution

In accordance with the general scheme described in section 2, to obtain the general solution for equations (3.4), (3.5) one should start with the mappings γ_{\pm} taking values in the subgroup \tilde{H} and satisfying relations (2.9). Write these mappings in the block matrix form

$$\gamma_{\pm} = \begin{pmatrix} \beta_{\pm 1} & 0 \\ 0 & \beta_{\pm 2} \end{pmatrix}$$

where the mappings $\beta_{\pm 1}$ take values in the group $\text{GL}(m_1, \mathbb{C})$, and the mappings $\beta_{\pm 2}$ take values in the group $\text{GL}(m_2, \mathbb{C})$. Now we have to integrate equations (2.10). Since almost all solutions can be obtained using the mappings μ_{\pm} taking values in the subgroups \tilde{N}_{\pm} , we choose for μ_{\pm} the parametrisation of the form

$$\mu_{-} = \begin{pmatrix} I_{m_1} & 0 \\ \mu_{-21} & I_{m_2} \end{pmatrix}, \quad \mu_{+} = \begin{pmatrix} I_{m_1} & \mu_{+12} \\ 0 & I_{m_2} \end{pmatrix}, \quad (3.11)$$

with the mapping μ_{-21} taking values in the space of $m_2 \times m_1$ matrices and the mapping μ_{+12} taking values in the space of $m_1 \times m_2$ matrices. This representation allows to reduce equations (2.10) to the equations

$$\begin{aligned} \partial_{-}\mu_{-21} &= \beta_{-2}C_{-}\beta_{-1}^{-1}, & \partial_{+}\mu_{-21} &= 0, \\ \partial_{+}\mu_{+12} &= \beta_{+1}C_{+}\beta_{+2}^{-1}, & \partial_{-}\mu_{+12} &= 0. \end{aligned}$$

The general solution to these equations is

$$\mu_{-21}(z^{-}) = m_{-21} + \int_0^{z^{-}} dy^{-} \beta_{-2}(y^{-}) C_{-} \beta_{-1}^{-1}(y^{-}), \quad (3.12)$$

$$\mu_{+12}(z^{+}) = m_{+12} + \int_0^{z^{+}} dy^{+} \beta_{+1}(y^{+}) C_{+} \beta_{+2}^{-1}(y^{+}), \quad (3.13)$$

where m_{-21} and m_{+12} are arbitrary $m_2 \times m_1$ and $m_1 \times m_2$ matrices respectively.

Consider now the Gauss decomposition (2.12). From (3.11) one gets

$$\mu_{+}^{-1} \mu_{-} = \begin{pmatrix} I_{m_1} - \mu_{+12} \mu_{-21} & -\mu_{+12} \\ \mu_{-21} & I_{m_2} \end{pmatrix}.$$

Parametrising the mapping η as

$$\eta = \begin{pmatrix} \eta_{11} & 0 \\ 0 & \eta_{22} \end{pmatrix},$$

and using (A.4), (A.5) we find that

$$\eta_{11} = I_{m_1} - \mu_{+12} \mu_{-21}, \quad \eta_{22} = I_{m_2} + \mu_{-21} (I_{m_1} - \mu_{+12} \mu_{-21})^{-1} \mu_{+12}.$$

Note that the mapping $\mu_{+}^{-1} \mu_{-}$ has the Gauss decomposition (2.12) only at those points of M for which

$$\det(I_{m_1} - \mu_{+12}(z^{+}) \mu_{-21}(z^{-})) \neq 0.$$

Now, using (2.13), we come to the following expression for the general solution of equations (3.4), (3.5):

$$\beta_1 = \beta_{+1}^{-1}(I_{m_1} - \mu_{+12}\mu_{-21})\beta_{-1}, \quad (3.14)$$

$$\beta_2 = \beta_{+2}^{-1} [I_{m_2} + \mu_{-21}(I_{m_1} - \mu_{+12}\mu_{-21})^{-1}\mu_{+12}] \beta_{-2}, \quad (3.15)$$

where the mappings μ_{-21} and μ_{+12} are given by (3.12) and (3.13).

Proceed now to the case when the considered \mathbb{Z} -gradation is associated with the $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra defined by (3.7). Recall that with the choice $c_{\pm} = x_{\pm}$ the Toda equations have form (3.8), (3.9). Introduce the notation $\beta = \beta_1$, and write expression (3.14) as

$$\beta(z^-, z^+) = \zeta_{+1}(z^+)\zeta_{-1}(z^-) + \zeta_{+2}(z^+)\zeta_{-2}(z^-),$$

where

$$\begin{aligned} \zeta_{+1} &= \beta_{+1}^{-1}, & \zeta_{-1} &= \beta_{-1}, \\ \zeta_{+2} &= -\beta_{+1}^{-1}\mu_{+12}, & \zeta_{-2} &= \mu_{-21}\beta_{-1}. \end{aligned}$$

From (3.8) one obtains

$$\beta_2 = -\partial_+\partial_-\beta + \partial_+\beta\beta^{-1}\partial_-\beta.$$

Consider the family of quasideterminants

$$\Delta_i = \left| \begin{array}{cccc} \beta & \partial_-\beta & \cdots & \partial_-^{i-1}\beta \\ \partial_+\beta & \partial_+\partial_-\beta & \cdots & \partial_+\partial_-^{i-1}\beta \\ \vdots & \vdots & \ddots & \vdots \\ \partial_+^{i-1}\beta & \partial_+^{i-1}\partial_-\beta & \cdots & \partial_+^{i-1}\partial_-^{i-1}\beta \end{array} \right|_{ii}. \quad (3.16)$$

The definition of a quasideterminant which is used in our work is given by (A.16). A more general definition of quasideterminants and investigation of their properties can be found in [9, 10]. It follows from (A.17) that

$$\beta_1 = \Delta_1, \quad \beta_2 = -\Delta_2.$$

It is another form of writing the general solution to equations (3.8), (3.9).

3.3 Generalisation

In this section we consider the Toda equations also based on the Lie group $GL(r+1, \mathbb{C})$ but with more general \mathbb{Z} -gradations of $\mathfrak{gl}(r+1, \mathbb{C})$. Strictly speaking, the systems under consideration are not maximally nonabelian Toda systems. Nevertheless, it is possible to find for their general solution some interesting explicit expression. Moreover, maximally nonabelian Toda systems based on classical Lie groups different from $GL(r+1, \mathbb{C})$, can be considered as reductions of the systems which are investigated here.

Let d_1 and d_2 be two positive integers such that $1 \leq d_1 < d_2 \leq r$. Consider the \mathbb{Z} -gradation of $\mathfrak{gl}(r+1, \mathbb{C})$ corresponding to the case when we choose all the labels of the Dynkin diagram equal to zero, except the labels s_{d_1} and s_{d_2} which are chosen equal to 1. The corresponding grading operator can be constructed as follows. Denote the

grading operator given by relation (3.2) by q_d , then it is clear that the grading operator q in question is $q = q_{d_1} + q_{d_2}$. The explicit form of q is

$$q = \begin{pmatrix} \frac{m_2+2m_3}{m_1+m_2+m_3}I_{m_1} & 0 & 0 \\ 0 & \frac{-m_1+m_3}{m_1+m_2+m_3}I_{m_2} & 0 \\ 0 & 0 & \frac{-2m_1-m_2}{m_1+m_2+m_3}I_{m_3} \end{pmatrix}, \quad (3.17)$$

where $m_1 = d_1$, $m_2 = d_2 - d_1$ and $m_3 = r + 1 - d_2$, so that $m_1 + m_2 + m_3 = r + 1$.

In this example we consider $(r+1) \times (r+1)$ matrices as 3×3 block matrices $x = (x_{ij})$ with x_{ij} being $m_i \times m_j$ matrices. With respect to the \mathbb{Z} -gradation of $\mathfrak{gl}(r+1, \mathbb{C})$ defined by the grading operator q given by (3.17), there are five nontrivial grading subspaces $\mathfrak{g}_{\pm 2}$, $\mathfrak{g}_{\pm 1}$ and \mathfrak{g}_0 . Here the subspace \mathfrak{g}_a is formed by the block matrices $x = (x_{ij})$ where only the blocks x_{ij} with $j - i = a$ are different from zero.

The subalgebra $\tilde{\mathfrak{h}}$ consists of all block diagonal matrices, and the subspaces $\tilde{\mathfrak{n}}_-$ and $\tilde{\mathfrak{n}}_+$ are formed by all block strictly lower and upper triangular matrices respectively. The subgroup \tilde{H} is formed by all block diagonal nondegenerate matrices, and the subgroups \tilde{N}_- and \tilde{N}_+ consist respectively of all block upper and lower triangular matrices with unit matrices on the diagonal.

In the case under consideration the general form of the elements $c_{\pm} \in \mathfrak{g}_{\pm 1}$ is

$$c_- = \begin{pmatrix} 0 & 0 & 0 \\ C_{-1} & 0 & 0 \\ 0 & C_{-2} & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_{+1} & 0 \\ 0 & 0 & C_{+2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Parametrising the mapping γ as

$$\gamma = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

we come to the following Toda equations

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}C_{+1}\beta_2C_{-1}, \quad (3.18)$$

$$\partial_+(\beta_2^{-1}\partial_-\beta_2) = -\beta_2^{-1}C_{+2}\beta_3C_{-2} + C_{-1}\beta_1^{-1}C_{+1}\beta_2, \quad (3.19)$$

$$\partial_+(\beta_3^{-1}\partial_-\beta_3) = C_{-2}\beta_2^{-1}C_{+2}\beta_3, \quad (3.20)$$

where the mappings β_i , $i = 1, 2, 3$, take values in the Lie groups $GL(m_i, \mathbb{C})$.

Find now the conditions which guarantee the existence of an $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra of $\mathfrak{gl}(r+1, \mathbb{C})$ giving the \mathbb{Z} -gradation under consideration. Actually we are interested only in an integral embedding of $\mathfrak{sl}(2, \mathbb{C})$. Therefore, the Chevalley generators x_- and x_+ should belong to the subspaces \mathfrak{g}_{-1} and \mathfrak{g}_{+1} respectively, and the Cartan generator h should coincide with $2q$. In this case relations (2.4) are satisfied. Write for x_- and x_+ the block matrix representation

$$x_- = \begin{pmatrix} 0 & 0 & 0 \\ X_{-1} & 0 & 0 \\ 0 & X_{-2} & 0 \end{pmatrix}, \quad x_+ = \begin{pmatrix} 0 & X_{+1} & 0 \\ 0 & 0 & X_{+2} \\ 0 & 0 & 0 \end{pmatrix}.$$

To satisfy relation (2.3) we should have

$$X_{+1}X_{-1} = 2\frac{m_2 + 2m_3}{m_1 + m_2 + m_3}I_{m_1}, \quad (3.21)$$

$$X_{+2}X_{-2} - X_{-1}X_{+1} = 2\frac{-m_1 + m_3}{m_1 + m_2 + m_3}I_{m_2}, \quad (3.22)$$

$$X_{-2}X_{+2} = 2\frac{2m_1 + m_2}{m_1 + m_2 + m_3}I_{m_3}. \quad (3.23)$$

From (3.21) it follows that $m_1 \leq m_2$ and that the ranks $r(X_{\pm 1})$ are equal to m_1 . On the other hand, relation (3.23) implies $m_3 \leq m_2$ and $r(X_{\pm 2}) = m_3$. Multiplying (3.22) from the left by X_{+1} and taking into account (3.21), one obtains

$$X_{+1}X_{+2}X_{-2} = 2\frac{-m_1 + m_2 + 3m_3}{m_1 + m_2 + m_3}X_{+1}.$$

This relation gives that $m_1 \leq m_3$. Similarly, multiplying (3.22) from the right by X_{+2} and taking into account (3.23), we get the relation

$$X_{-1}X_{+1}X_{+2} = 2\frac{3m_1 + m_2 - m_3}{m_1 + m_2 + m_3}X_{+2},$$

which implies that $m_3 \leq m_1$. Thus, one can satisfy relations (3.21)–(3.23) only if $m_1 = m_3$. It is easy to see that the conditions $m_1 = m_3$ and $m_1 \leq m_2$ are sufficient conditions for the existence of an $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra leading to the \mathbb{Z} -gradation under consideration.

In the simplest and most symmetric case arising when $m_1 = m_2 = m_3 = m$, we can take, without any loss of generality, as the sought for $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra the subalgebra generated by the elements

$$h = \begin{pmatrix} 2I_m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2I_m \end{pmatrix}, \quad (3.24)$$

$$x_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}I_m & 0 & 0 \\ 0 & \sqrt{2}I_m & 0 \end{pmatrix}, \quad x_+ = \begin{pmatrix} 0 & \sqrt{2}I_m & 0 \\ 0 & 0 & \sqrt{2}I_m \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.25)$$

With the choice $c_{\pm} = x_{\pm}/\sqrt{2}$ one comes to the Toda equations of the form

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}\beta_2, \quad (3.26)$$

$$\partial_+(\beta_2^{-1}\partial_-\beta_2) = -\beta_2^{-1}\beta_3 + \beta_1^{-1}\beta_2, \quad (3.27)$$

$$\partial_+(\beta_3^{-1}\partial_-\beta_3) = \beta_2^{-1}\beta_3, \quad (3.28)$$

where the mappings β_i , $i = 1, 2, 3$, take values in the Lie group $GL(m, \mathbb{C})$.

Return now to the case of arbitrary m_1 , m_2 and m_3 . To obtain the general solution of equations (3.18)–(3.20) we start with the mappings γ_{\pm} which are parametrised as

$$\gamma_{\pm} = \begin{pmatrix} \beta_{\pm 1} & 0 & 0 \\ 0 & \beta_{\pm 2} & 0 \\ 0 & 0 & \beta_{\pm 3} \end{pmatrix}.$$

Write for the mappings μ_{\pm} the following representation

$$\mu_- = \begin{pmatrix} I_{m_1} & 0 & 0 \\ \mu_{-21} & I_{m_2} & 0 \\ \mu_{-31} & \mu_{-32} & I_{m_3} \end{pmatrix}, \quad \mu_+ = \begin{pmatrix} I_{m_1} & \mu_{+12} & \mu_{+13} \\ 0 & I_{m_2} & \mu_{+23} \\ 0 & 0 & I_{m_3} \end{pmatrix}.$$

Then equations (2.10) take the form

$$\begin{aligned} \partial_- \mu_{-21} &= \beta_{-2} C_{-1} \beta_{-1}^{-1}, & \partial_+ \mu_{-21} &= 0, \\ \partial_- \mu_{-32} &= \beta_{-3} C_{-2} \beta_{-2}^{-1}, & \partial_+ \mu_{-32} &= 0, \\ \partial_- \mu_{-31} &= \mu_{-32} \beta_{-2} C_{-1} \beta_{-1}^{-1}, & \partial_+ \mu_{-31} &= 0, \\ \partial_+ \mu_{+12} &= \beta_{+1} C_{+1} \beta_{+2}^{-1}, & \partial_- \mu_{+12} &= 0, \\ \partial_+ \mu_{+23} &= \beta_{+2} C_{+2} \beta_{+3}^{-1}, & \partial_- \mu_{+23} &= 0, \\ \partial_+ \mu_{+13} &= \mu_{+12} \beta_{+2} C_{+2} \beta_{+3}^{-1}, & \partial_- \mu_{+13} &= 0. \end{aligned}$$

The general solution to these equations is

$$\begin{aligned} \mu_{-21}(z^-) &= m_{-21} + \int_0^{z^-} dy_1^- \beta_{-2}(y_1^-) C_{-1} \beta_{-1}^{-1}(y_1^-), \\ \mu_{-32}(z^-) &= m_{-32} + \int_0^{z^-} dy_1^- \beta_{-3}(y_1^-) C_{-2} \beta_{-2}^{-1}(y_1^-), \\ \mu_{-31}(z^-) &= m_{-31} \\ &+ \int_0^{z^-} dy_2^- \left(m_{-32} + \int_0^{y_2^-} dy_1^- \beta_{-3}(y_1^-) C_{-2} \beta_{-2}^{-1}(y_1^-) \right) \beta_{-2}(y_2^-) C_{-1} \beta_{-1}^{-1}(y_2^-), \\ \mu_{+12}(z^+) &= m_{+12} + \int_0^{z^+} dy_1^+ \beta_{+1}(y_1^+) C_{+1} \beta_{+2}^{-1}(y_1^+), \\ \mu_{+23}(z^+) &= m_{+23} + \int_0^{z^+} dy_1^+ \beta_{+2}(y_1^+) C_{+2} \beta_{+3}^{-1}(y_1^+), \\ \mu_{+13}(z^+) &= m_{+13} \\ &+ \int_0^{z^+} dy_2^+ \left(m_{+12} + \int_0^{y_2^+} dy_1^+ \beta_{+1}(y_1^+) C_{+1} \beta_{+2}^{-1}(y_1^+) \right) \beta_{+2}(y_2^+) C_{+2} \beta_{+3}^{-1}(y_2^+), \end{aligned}$$

where m_{-21} , m_{-32} , m_{-31} , m_{+12} , m_{+23} and m_{+13} are arbitrary constant matrices.

The next step of the integration procedure is to obtain from the Gauss decomposition (2.12) the mapping η . Using the relation

$$\mu_+^{-1} = \begin{pmatrix} I_{m_1} & -\mu_{+12} & -(\mu_{+13} - \mu_{+12}\mu_{+23}) \\ 0 & I_{m_2} & -\mu_{+23} \\ 0 & 0 & I_{m_3} \end{pmatrix},$$

we get for the blocks determining the mapping $\mu_+^{-1}\mu_-$ the following expressions

$$(\mu_+^{-1}\mu_-)_{11} = I_{m_1} - \mu_{+12}\mu_{-21} - (\mu_{+13} - \mu_{+12}\mu_{+23})\mu_{-31},$$

$$\begin{aligned}
(\mu_+^{-1}\mu_-)_{12} &= -\mu_{+12} - (\mu_{+13} - \mu_{+12}\mu_{+23})\mu_{-32}, \\
(\mu_+^{-1}\mu_-)_{13} &= -(\mu_{+13} - \mu_{+12}\mu_{+23}), \\
(\mu_+^{-1}\mu_-)_{21} &= \mu_{-21} - \mu_{+23}\mu_{-31}, \quad (\mu_+^{-1}\mu_-)_{22} = I_{m_2} - \mu_{+33}\mu_{-32}, \\
(\mu_+^{-1}\mu_-)_{23} &= -\mu_{+23}, \quad (\mu_+^{-1}\mu_-)_{31} = \mu_{-31}, \\
(\mu_+^{-1}\mu_-)_{32} &= \mu_{-32}, \quad (\mu_+^{-1}\mu_-)_{33} = I_{m_3}.
\end{aligned}$$

Now, using relations (A.10)–(A.12) one can write down the expressions for the mappings η_{11} , η_{22} and η_{33} entering the parametrisation of the mapping η ,

$$\eta = \begin{pmatrix} \eta_{11} & 0 & 0 \\ 0 & \eta_{22} & 0 \\ 0 & 0 & \eta_{33} \end{pmatrix}.$$

The corresponding expressions are rather cumbersome, so we give here only one for η_{11} ,

$$\eta_{11} = I_{m_1} - \mu_{+12}\mu_{-21} - (\mu_{+13} - \mu_{+12}\mu_{+23})\mu_{-31}.$$

Finally, relation (2.13) allows us to write the general solution of Toda equations (3.18)–(3.20) in an explicit form. In particular, the expression for the mapping β_1 is

$$\beta_1 = \beta_{+1}^{-1}(I_{m_1} - \mu_{+12}\mu_{-21} - (\mu_{+13} - \mu_{+12}\mu_{+23})\mu_{-31})\beta_{-1}. \quad (3.29)$$

Consider now the case $m_1 = m_2 = m_3 = m$. Recall that in this case the \mathbb{Z} -gradation under consideration can be associated with the $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra generated by the elements h and x_{\pm} defined by (3.24) and (3.25). Choosing again $c_{\pm} = x_{\pm}/\sqrt{2}$, we come to the Toda equations (3.26)–(3.28). Denote $\beta = \beta_1$, and write β in the following form

$$\beta(z^-, z^+) = \zeta_{+1}(z^+)\zeta_{-1}(z^-) + \zeta_{+2}(z^+)\zeta_{-2}(z^-) + \zeta_{+3}(z^+)\zeta_{-3}(z^-)$$

which follows from (3.29). Here

$$\begin{aligned}
\zeta_{+1} &= \beta_{+1}^{-1}, & \zeta_{-1} &= \beta_{-1}, \\
\zeta_{+2} &= -\beta_{+1}^{-1}\mu_{+12}, & \zeta_{-2} &= \mu_{-21}\beta_{-1}, \\
\zeta_{+3} &= -\beta_{+1}^{-1}(\mu_{+13} - \mu_{+12}\mu_{+23}), & \zeta_{-3} &= \mu_{-31}\beta_{-1}.
\end{aligned}$$

Using equations (3.26) and (3.27), one obtains

$$\beta_1 = \Delta_1, \quad \beta_2 = -\Delta_2, \quad \beta_3 = \Delta_3,$$

where the quasideterminants Δ_i are defined by (3.16).

It is interesting to go further and consider the case when $r + 1 = pm$, where p and m are positive integers. Suppose that $s_m = s_{2m} = \dots = s_{(p-1)m} = 1$, and all remaining labels of the Dynkin diagram are equal to zero. The corresponding grading operator can be written as a $p \times p$ block matrix $q = (q_{ij})$ with

$$q_{ij} = \frac{1}{2}(p + 1 - 2i)\delta_{ij}I_m.$$

The corresponding \mathbb{Z} -gradation of the Lie algebra $\mathfrak{gl}(r+1, \mathbb{C})$ can be associated with the $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra generated by the elements $h = (h_{ij})$, $x_- = (x_{-ij})$ and $x_+ = (x_{+ij})$, where

$$h_{ij} = (p+1-2i)\delta_{ij}I_m, \\ x_{-ij} = \sqrt{i(p-i)}\delta_{i,j+1}I_m, \quad x_{+ij} = \sqrt{i(p-i)}\delta_{i+1,j}I_m.$$

Defining the elements c_- and c_+ by the relations

$$c_{-ij} = \delta_{i,j+1}I_m, \quad c_{+ij} = \delta_{i+1,j}I_m,$$

one comes to the following Toda equations

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}\beta_2, \quad (3.30)$$

$$\partial_+(\beta_i^{-1}\partial_-\beta_i) = -\beta_i^{-1}\beta_{i+1} + \beta_{i-1}^{-1}\beta_i, \quad 1 < i < p, \quad (3.31)$$

$$\partial_+(\beta_p^{-1}\partial_-\beta_p) = \beta_{p-1}^{-1}\beta_p. \quad (3.32)$$

Denote $\beta = \beta_1$, then one can show that from equations (3.30)–(3.32) it follows that

$$\beta_i = (-1)^{i-1}\Delta_i, \quad (3.33)$$

where the quasideterminants Δ_i are defined by (3.16). On the other hand, analysing the structure of the Gauss decomposition of block matrices, we conclude that β can be represented as

$$\beta(z^-, z^+) = \sum_{i=1}^p \zeta_{+i}(z^+) \zeta_{-i}(z^-), \quad (3.34)$$

where the mappings $\zeta_{\pm i}$ take values in the space of $m \times m$ matrices. Using arbitrary mappings $\zeta_{\pm i}$, we obtain the general solution to system (3.30)–(3.32). A proof of relation (3.33) is given in appendix B.

Slightly modifying the consideration given in paper [10], one can state the following. Consider the one dimensional infinite system of ordinary differential equations

$$\frac{d}{dt} \left(\beta_1^{-1} \frac{d\beta_1}{dt} \right) = -\beta_1^{-1}\beta_2, \\ \frac{d}{dt} \left(\beta_i^{-1} \frac{d\beta_i}{dt} \right) = -\beta_i^{-1}\beta_{i+1} + \beta_{i-1}^{-1}\beta_i, \quad i > 1,$$

where β_i , $i = 1, 2, \dots$, are $\text{GL}(m, \mathbb{C})$ valued functions of the real variable t . Denoting $\beta = \beta_1$, one can write the general solution to this system as

$$\beta_i = (-1)^{i-1}\Gamma_i, \quad (3.35)$$

where Γ_i , $i = 1, 2, \dots$, are the quasideterminants defined by

$$\Gamma_i = \left| \begin{array}{cccc} \beta & \frac{d\beta}{dt} & \dots & \frac{d^{i-1}\beta}{dt^{i-1}} \\ \frac{d\beta}{dt} & \frac{d^2\beta}{dt^2} & \dots & \frac{d^i\beta}{dt^i} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{i-1}\beta}{dt^{i-1}} & \frac{d^i\beta}{dt^i} & \dots & \frac{d^{2i-2}\beta}{dt^{2i-2}} \end{array} \right|_{ii}.$$

Hence, relation (3.33) can be considered as a two dimensional generalisation of (3.35). Actually, relation (3.33) for arbitrary β gives the general solution to the infinite system of equations having form (3.30) and (3.31) without the condition $i < p$. Here if β has form (3.34), then one gets $\Delta_{p+1} = 0$ and comes to the general solution of the finite system (3.30)–(3.32). Note that the expression for β_1 in form $\beta_1 = \beta$ with β given by (3.34), was also obtained in paper [11] by some other method.

4 Complex orthogonal group

The complex orthogonal group $O(n, \mathbb{C})$ is the Lie subgroup of the Lie group $GL(n, \mathbb{C})$ formed by matrices $a \in GL(n, \mathbb{C})$ satisfying the condition

$$\tilde{I}_n a^t \tilde{I}_n = a^{-1}, \quad (4.1)$$

where \tilde{I}_n is the antidiagonal unit $n \times n$ matrix, and a^t is the transpose of a . The corresponding Lie algebra $\mathfrak{o}(n, \mathbb{C})$ is the subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ which consists of the matrices x satisfying the condition

$$\tilde{I}_n x^t \tilde{I}_n = -x. \quad (4.2)$$

For an $m_1 \times m_2$ matrix a we will denote by a^T the matrix defined by the relation

$$a^T = I_{m_2} a^t I_{m_1}.$$

Using this notation, we can rewrite conditions (4.1) and (4.2) as $a^T = a^{-1}$ and $x^T = -x$. The Lie algebra $\mathfrak{o}(n, \mathbb{C})$ is simple. For $n = 2r + 1$ it is of type B_r , while for $n = 2r$ it is of type D_r . Discuss these two cases separately.

Consider the \mathbb{Z} -gradation of $\mathfrak{o}(2r + 1, \mathbb{C})$ arising when we choose $s_d = 1$ for some fixed d such that $1 \leq d \leq r$, and put all other labels of the Dynkin diagram be equal to zero. Using relation (2.2), one gets

$$\begin{aligned} q &= \sum_{i=1}^{r-1} i h_i + \frac{1}{2} r h_r, \quad d = r, \\ q &= \sum_{i=1}^{r-1} i h_i + \frac{1}{2} (r-1) h_r, \quad d = r-1, \\ q &= \sum_{i=1}^d i h_i + d \sum_{i=d+1}^{r-1} h_i + \frac{1}{2} d h_r, \quad 1 \leq d < r-1. \end{aligned}$$

It is convenient to choose the following Cartan generators of $\mathfrak{o}(2r + 1, \mathbb{C})$:

$$\begin{aligned} h_i &= e_{i,i} - e_{i+1,i+1} + e_{2r+1-i,2r+1-i} - e_{2r+2-i,2r+2-i}, \quad 1 \leq i < r, \\ h_r &= 2(e_{r,r} - e_{r+2,r+2}), \end{aligned}$$

where the matrices $e_{i,j}$ are defined by (3.1). Using these expressions one obtains

$$q = \sum_{i=1}^d e_{i,i} - \sum_{i=1}^d e_{2r+2-i,2r+2-i}.$$

Denoting $m_1 = d$ and $m_2 = 2(r - d) + 1$, we write q in the block matrix form,

$$q = \begin{pmatrix} I_{m_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{m_1} \end{pmatrix}, \quad (4.3)$$

where zero on the diagonal stands for the $m_2 \times m_2$ block of zeros.

It is easy to verify that in the case of the Lie algebra $\mathfrak{o}(2r, \mathbb{C})$

$$q = \frac{1}{2} \sum_{i=1}^{r-2} i h_i + \frac{1}{4}(r-2)h_{r-1} + \frac{1}{4}r h_r, \quad d = r,$$

$$q = \frac{1}{2} \sum_{i=1}^{r-2} i h_i + \frac{1}{4}r h_{r-1} + \frac{1}{4}(r-2)h_r, \quad d = r-1,$$

$$q = \sum_{i=1}^d i h_i + d \sum_{i=d+1}^{r-2} h_i + \frac{1}{2}d(h_{r-1} + h_r), \quad 1 \leq d < r-1.$$

Choose as the Cartan generators of $\mathfrak{o}(2r, \mathbb{C})$ the elements

$$h_i = e_{i,i} - e_{i+1,i+1} + e_{2r-i,2r-i} - e_{2r+1-i,2r+1-i}, \quad 1 \leq i < r,$$

$$h_r = e_{r-1,r-1} + e_{r,r} - e_{r+1,r+1} - e_{r+2,r+2}.$$

Then one easily obtains

$$q = \frac{1}{2} \sum_{i=1}^r e_{i,i} - \frac{1}{2} \sum_{i=1}^r e_{2r+1-i,2r+1-i}, \quad d = r,$$

$$q = \frac{1}{2} \sum_{i=1}^{r-1} e_{i,i} - \frac{1}{2}e_{r,r} + \frac{1}{2}e_{r+1,r+1} - \frac{1}{2} \sum_{i=1}^{r-1} e_{2r+1-i,2r+1-i}, \quad d = r-1,$$

$$q = \sum_{i=1}^d e_{i,i} - \sum_{i=1}^d e_{2r+1-i,2r+1-i}, \quad 1 \leq i < r-1.$$

Note that the grading operators corresponding to the cases $d = r$ and $d = r-1$ are connected by the automorphism σ of $\mathfrak{o}(2r, \mathbb{C})$ defined by the relation $\sigma(x) = axa^{-1}$, where a is the matrix corresponding to the permutation of the indices r and $r+1$. There is the corresponding automorphism of the Lie group $O(2r, \mathbb{C})$, which is defined by the same formula. Thus, the cases $d = r$ and $d = r-1$ leads actually to the same Toda equations, and we can exclude one of them, for example $d = r-1$ from the consideration.

For the case $d = r$ the grading operator has the following block form

$$q = \frac{1}{2} \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}, \quad (4.4)$$

where we denoted $m = r$. In the case $1 \leq d < r-2$ denoting $m_1 = d$ and $m_2 = 2(r-d)$ one sees that the grading operator q has form (4.3).

Resuming the consideration, we can say that for any representation of the positive integer n in the form $n = 2m_1 + m_2$ where m_1 and m_2 are positive integers such that $m_2 \neq 2$, there is the \mathbb{Z} -gradation of $\mathfrak{o}(n, \mathbb{C})$ corresponding to a maximally nonabelian Toda system. This gradation is generated by the grading operator (4.3). In the case $n = 2m$, there is one more \mathbb{Z} -gradation defined by the grading operator (4.4).

Probably, the absence of the case $m_2 = 2$ requires a special explanation. Actually the operator q given by (4.3) with $m_2 = 2$ defines some \mathbb{Z} -gradation of the corresponding complex orthogonal algebra, but this gradation corresponds to the case when two labels of the Dynkin diagram, s_{r-1} and s_r , are equal to 1. Therefore, the corresponding Toda system is not maximally abelian. Nevertheless, it is convenient to consider the case $m_2 = 2$ together with those which do correspond to maximally nonabelian systems.

The \mathbb{Z} -gradation defined by the grading operator (4.4) can be associated with an $\mathrm{SL}(2, \mathbb{C})$ -subalgebra of the Lie algebra $\mathfrak{o}(2m, \mathbb{C})$ if and only if the integer m is even. In this case the corresponding element h coincides with $2q$, and the elements x_{\pm} have form (3.6), where the matrices X_{\pm} satisfy the conditions

$$X_+ X_- = I_m, \quad X_{\pm}^T = -X_{\pm}.$$

With a \mathbb{Z} -gradation generated by the grading operator of form (4.3) one can find the corresponding $\mathrm{SL}(2, \mathbb{C})$ -subalgebra of $\mathfrak{o}(n, \mathbb{C})$ if and only if $m_1 \leq m_2$. Here $h = 2q$ and the elements x_{\pm} are

$$x_- = \begin{pmatrix} 0 & 0 & 0 \\ X_- & 0 & 0 \\ 0 & -X_-^T & 0 \end{pmatrix}, \quad x_+ = \begin{pmatrix} 0 & X_+ & 0 \\ 0 & 0 & -X_+^T \\ 0 & 0 & 0 \end{pmatrix},$$

where the matrices X_{\pm} satisfy the conditions

$$X_+ X_- = 2I_{m_1}, \quad X_+^T X_-^T - X_- X_+ = 0.$$

Proceed now to the consideration of the Toda equations associated to the gradations described above. Begin with the gradation generated by the grading operator (4.4). The general form of the elements $c_{\pm} \in \mathfrak{g}_{\pm 1}$ is given by (3.3), where the matrices C_{\pm} should satisfy the relations

$$C_{\pm}^T = -C_{\pm}. \quad (4.5)$$

The subgroup \tilde{H} in the case under consideration is formed by the 2×2 block matrices of the form

$$h = \begin{pmatrix} h_{11} & 0 \\ 0 & (h_{11}^{-1})^T \end{pmatrix},$$

where $h_{11} \in \mathrm{GL}(m, \mathbb{C})$. Hence, the mapping γ has the following block form

$$\gamma = \begin{pmatrix} \beta & 0 \\ 0 & (\beta^{-1})^T \end{pmatrix}, \quad (4.6)$$

where the mapping β take values in the Lie group $\mathrm{GL}(m, \mathbb{C})$. The corresponding Toda equations are

$$\partial_+(\beta^{-1} \partial_- \beta) = -\beta^{-1} C_+ (\beta^{-1})^T C_-. \quad (4.7)$$

It is clear that these equations can be considered as the result of the reduction of equations (3.4), (3.5) to the case $\beta_2 = (\beta_1^{-1})^T$, which is possible if relations (4.5) are valid. Therefore, the general solution to equation (4.7) can be obtained from the general solution to equations (3.4), (3.5) by the method described in [4].

Certainly, we can get the general solution directly, using the general scheme of section 2.4. The mappings γ_{\pm} in the case under consideration have the form

$$\gamma_{\pm} = \begin{pmatrix} \beta_{\pm} & 0 \\ 0 & (\beta_{\pm}^{-1})^T \end{pmatrix},$$

where the mappings β_{\pm} take values in the Lie group $\text{GL}(m, \mathbb{C})$. Note that the subgroups \tilde{N}_- and \tilde{N}_+ in the case under consideration are formed respectively by the block matrices

$$n_- = \begin{pmatrix} I_m & 0 \\ n_{-21} & I_m \end{pmatrix}, \quad n_+ = \begin{pmatrix} I_m & n_{+12} \\ 0 & I_m \end{pmatrix},$$

where the matrices n_{-21} and n_{+12} obey the equalities

$$n_{-21}^T = -n_{-21}, \quad n_{+12}^T = -n_{+12}.$$

Representing the mappings μ_{\pm} as

$$\mu_- = \begin{pmatrix} I_m & 0 \\ \mu_{-21} & I_m \end{pmatrix}, \quad \mu_+ = \begin{pmatrix} I_m & \mu_{+12} \\ 0 & I_m \end{pmatrix},$$

one obtains

$$\mu_{-21}(z^-) = m_{-21} + \int_0^{z^-} dy^- (\beta_-^{-1})^T(y^-) C_- \beta_-^{-1}(y^-), \quad (4.8)$$

$$\mu_{+12}(z^+) = m_{+12} + \int_0^{z^+} dy^+ \beta_+(y^+) C_+ \beta_+^T(y^+), \quad (4.9)$$

where the constant matrices m_{-21} and m_{+12} satisfy the relations

$$m_{-21}^T = -m_{-21}, \quad m_{+12}^T = -m_{+12}.$$

The final expression for the general solution to equations (4.7) is

$$\beta = \beta_+^{-1} (I_m - \mu_{+12} \mu_{-21}) \beta_-, \quad (4.10)$$

where the mappings μ_{-21} and μ_{+12} are given by (4.8) and (4.9).

Consider now the Toda equations arising when we choose the \mathbb{Z} -gradation of $\mathfrak{o}(n, \mathbb{C})$ generated by the grading operator q defined by (4.3). In this case the general form of the elements c_{\pm} is

$$c_- = \begin{pmatrix} 0 & 0 & 0 \\ C_- & 0 & 0 \\ 0 & -C_-^T & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_+ & 0 \\ 0 & 0 & -C_+^T \\ 0 & 0 & 0 \end{pmatrix}.$$

The mapping γ has the following block form

$$\gamma = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & (\beta_1^{-1})^T \end{pmatrix}, \quad (4.11)$$

where the mappings β_1 and β_2 take values in the Lie groups $\text{GL}(m_1, \mathbb{C})$ and $\text{O}(m_2, \mathbb{C})$ respectively. The corresponding Toda equations are

$$\begin{aligned} \partial_+(\beta_1^{-1}\partial_-\beta_1) &= -\beta_1^{-1}C_+\beta_2C_-, \\ \partial_+(\beta_2^{-1}\partial_-\beta_2) &= -(C_-\beta_1^{-1}C_+\beta_2)^T + C_-\beta_1^{-1}C_+\beta_2. \end{aligned}$$

These equations can be considered as the reduction of equations (3.18)–(3.20) to the case $\beta_3 = \beta_1$ and $\beta_2^T = \beta_2^{-1}$. Such a reduction is possible if the matrices $C_{\pm 1}$ and $C_{\pm 2}$ in (3.18)–(3.20) satisfy the conditions

$$C_{-1} = -C_{-2}^T = C_-, \quad C_{+1} = -C_{+2}^T = C_+.$$

The general solution to equations (4.12), (4.12) can be obtained either by the reduction of the general solution to equations (3.18)–(3.20), or directly. The corresponding expressions are rather cumbersome and we do not give them here. Note only that the corresponding subgroups \tilde{N}_- and \tilde{N}_+ are composed respectively from the block matrices of the form

$$n_- = \begin{pmatrix} I_{m_1} & 0 & 0 \\ n_{-21} & I_{m_2} & 0 \\ n_{-31} & n_{-32} & I_{m_1} \end{pmatrix}, \quad n_+ = \begin{pmatrix} I_{m_1} & n_{+12} & n_{+13} \\ 0 & I_{m_2} & n_{+23} \\ 0 & 0 & I_{m_1} \end{pmatrix}. \quad (4.12)$$

where

$$\begin{aligned} n_{+23}^T &= -n_{+12}, & n_{+13}^T &= -n_{+13} + n_{+12}n_{+23}, \\ n_{-31}^T &= -n_{-21}, & n_{-32}^T &= -n_{-31} + n_{-32}n_{-21}. \end{aligned}$$

The maximally nonabelian Toda system based on the Lie group $\text{O}(5, \mathbb{C})$ and with the choice $d = 1$ in relation to the physics of black holes was investigated in paper [12], see also [13]. In these papers there was used a local parametrisation of the Lie group \tilde{H} , which is isomorphic here to $\text{GL}(1, \mathbb{C}) \times \text{O}(3, \mathbb{C})$. With the approach developed in the present paper, one can write the general solution in terms of corresponding matrices without using any local coordinates. Actually, for the system considered in [12] and [13] it is easier to use the fact that the Lie group $\text{O}(5, \mathbb{C})$ is locally isomorphic to the Lie group $\text{Sp}(4, \mathbb{C})$ and to consider the corresponding maximally nonabelian Toda system based on $\text{Sp}(4, \mathbb{C})$. It will be done in the next section.

5 Complex symplectic group

We define the complex symplectic group $\text{Sp}(2r, \mathbb{C})$ as the Lie subgroup of the Lie group $\text{GL}(2r, \mathbb{C})$ which consists of the matrices $a \in \text{GL}(2r, \mathbb{C})$ satisfying the condition

$$\tilde{J}_r a^t \tilde{J}_r = -a^{-1},$$

where \tilde{J}_r is the matrix given by

$$\tilde{J}_r = \begin{pmatrix} 0 & \tilde{I}_r \\ -\tilde{I}_r & 0 \end{pmatrix}.$$

The corresponding Lie algebra $\mathfrak{sp}(r, \mathbb{C})$ is defined as the subalgebra of the Lie algebra $\mathfrak{sl}(2r, \mathbb{C})$ formed by the matrices x which satisfy the condition

$$\tilde{J}_r x^t \tilde{J}_r = x.$$

The Lie algebra $\mathfrak{sp}(r, \mathbb{C})$ is simple, and it is of type C_r . Therefore, the Cartan matrix of $\mathfrak{sp}(r, \mathbb{C})$ is the transpose of the Cartan matrix of $\mathfrak{o}(n, \mathbb{C})$, and the same is for inverse of the Cartan matrix of $\mathfrak{sp}(r, \mathbb{C})$. For any fixed integer d such that $1 \leq d \leq r$, consider the \mathbb{Z} -gradation of $\mathfrak{sp}(r, \mathbb{C})$ arising when we choose all the labels of the corresponding Dynkin diagram equal to zero, except the label s_d , which we choose to be equal to 1. Using relation (2.2), we obtain the following expressions for the grading operator,

$$q = \frac{1}{2} \sum_{i=1}^r ih_i, \quad d = r, \quad q = \sum_{i=1}^d ih_i + d \sum_{i=d+1}^r h_i, \quad 1 \leq d < r.$$

Using the following choice of the Cartan generators,

$$\begin{aligned} h_i &= e_{i,i} - e_{i+1,i+1} + e_{2r-i,2r-i} - e_{2r+1-i,2r+1-i}, \quad 1 \leq i < d, \\ h_r &= e_{r,r} - e_{r+1,r+1}, \end{aligned}$$

one sees that the grading operator for the case $d = r$ has form (4.4) with $m = r$, and for the case $1 \leq d < r$ it has form (4.3) with $m_1 = d$ and $m_2 = 2(r - d)$. The corresponding $\mathrm{SL}(2, \mathbb{C})$ -subalgebra always exists for the case $d = r$, and for the case $1 \leq d < r$ it exists if and only if $m_1 \leq m_2$.

In the case $d = r$ the general form of the elements c_{\pm} is given by (3.3), where the matrices C_{\pm} satisfy the conditions

$$C_{\pm}^T = C_{\pm}. \quad (5.1)$$

The mapping γ has here form (4.6) and the Toda equations coincide with (4.7) where the matrices C_{\pm} satisfy (5.1). The obtained equations can be considered as the reduction of equations (3.4), (3.5) to the case $\beta_1 = (\beta_2^{-1})^T = \beta$ which is possible when (5.1) is valid. The general solution of the Toda equations is described by relation (4.10) where the mappings μ_{-21} and μ_{+12} are given by (4.8) and (4.9) with the constant matrices m_{-21} and m_{+12} satisfying the relations

$$m_{-21}^T = m_{-21}, \quad m_{+12}^T = m_{+12}.$$

The simplest choice of the matrices C_{\pm} is $C_{\pm} = I_r$. Here the Toda equations take the form

$$\partial_+(\beta^{-1}\partial_-\beta) = -(\beta^T\beta)^{-1}.$$

The subgroups \tilde{H}_{\pm} determining the symmetry transformations (2.8) are isomorphic to the Lie group $\mathrm{O}(r, \mathbb{C})$ and are composed of the matrices of form (3.10), where the matrix h_{11} satisfy the condition $h_{11}^T = h_{11}^{-1}$. With $C_{\pm} = \tilde{I}_r$ we come to the equations

$$\partial_+(\beta^{-1}\partial_-\beta) = -(\beta^t\beta)^{-1}.$$

Return to the discussion given in the end of the previous section. It is clear that choosing consistent local parametrisation for the Lie groups $\text{Sp}(4, \mathbb{C})$ and $\text{O}(5, \mathbb{C})$, we can obtain from the general solution of the Toda equations for $\text{Sp}(4, \mathbb{C})$ the general solution for the corresponding Toda equations based on $\text{O}(5, \mathbb{C})$. It can be verified that this solution coincides with the solution obtained in [13].

Proceed now to the case $1 \leq d < r$. In this case the general form of the elements c_{\pm} is

$$c_- = \begin{pmatrix} 0 & 0 & 0 \\ C_- & 0 & 0 \\ 0 & -\tilde{I}_d C_-^t \tilde{J}_{r-d} & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_+ & 0 \\ 0 & 0 & \tilde{J}_{r-d} C_+^t \tilde{I}_d \\ 0 & 0 & 0 \end{pmatrix};$$

the mapping γ has form (4.11) where the mappings β_1 and β_2 take values in the Lie groups $\text{GL}(d, \mathbb{C})$ and $\text{Sp}(2(r-d), \mathbb{C})$ respectively; and the Toda equations are

$$\partial_+(\beta_1^{-1} \partial_- \beta_1) = -\beta_1^{-1} C_+ \beta_2 C_-, \quad (5.2)$$

$$\partial_+(\beta_2^{-1} \partial_- \beta_2) = \beta_2^{-1} \tilde{J}_{r-d} C_+^t (\beta_1^{-1})^t C_-^t \tilde{J}_{r-d} + C_- \beta_1^{-1} C_+ \beta_2. \quad (5.3)$$

These equations are the reduction of equations (3.18)–(3.20) to the case $\beta_3 = \beta_1$ and $\tilde{J}_{r-d} \beta_2^t \tilde{J}_{r-d} = -\beta_2^{-1}$, which is possible if the matrices $C_{\pm 1}$ and $C_{\pm 2}$ in (3.18)–(3.20) satisfy the conditions

$$C_{-1} = -\tilde{J}_{r-d} C_{-2}^t \tilde{I}_d = C_-, \quad C_{+1} = \tilde{I}_d C_{+2}^t \tilde{J}_{r-d} = C_+.$$

It is quite clear that using the integration procedure described in section 2.4, one can easily write the general solution for equations (5.2), (5.3).

6 Concluding remarks

The main goal of our study was to obtain explicit expressions for the general solutions for some class of nonabelian Toda systems, namely for the maximally nonabelian ones, which has a very simple structure. Our consideration concerns finite dimensional complex semisimple Lie groups, but can be extended for the exceptional groups and infinite dimensional loop groups. In particular, starting with the loop group associated with the complex general group it is possible to come to the periodic two dimensional nonabelian Toda chain which was obtained by H. W. Capel and J. H. H. Perk in the context of the inhomogeneous XY -model [15], and also by A. V. Mikhailov in the framework of the reduction scheme [16]. Finite-gap solutions to this equations were found by I. M. Krichever [14].

Actually one can consider the infinite system of equations of form (3.31) with $i = 0, \pm 1, \pm 2, \dots$. This system was introduced by A. M. Polyakov at the end of seventies. The finite two dimensional nonabelian Toda chain and the periodic one can be considered as special cases of such system corresponding to the boundary conditions $\beta_0^{-1} = \beta_{p+1} = 0$ and $\beta_0 = \beta_{p+1}$ respectively.

We believe that nonabelian Toda systems are quite relevant for a number of applications in theoretical and mathematical physics, especially in particle and statistical physics, and in a near future their role for the description of nonlinear phenomena in these areas will be not less than that for the abelian systems.

Finishing up the paper, we would like to mention about one more reason why non-abelian Toda systems are not yet popular and known enough in mathematics, in particular in algebraic and differential geometry. The problem is that such important notions as the Gauss-Manin flat connection, the Griffiths transversality of variations of the Hodge structures, superhorizontal distributions, local and global Plücker relations, etc., which are perfectly fitted in the scheme with abelian Toda system, need to be re-understood or extended for nonabelian case. The same is for the systems generated by flat connections with values in higher grading subspaces of complex \mathbb{Z} -graded Lie algebra. We hope that the progress in this direction will be very fruitful for studies in theoretical physics as well as for mathematics itself.

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Note added in proof

After our paper was submitted to the journal there appeared an electronic preprint of P. Etingof, I. Gelfand and V. Retakh [17] concerning in particular nonabelian Toda systems. In this interesting paper some of our results are reproduced by some other method.

A Gauss decomposition of block matrices

Let $a = (a_{ij})$ be a nondegenerate block matrix formed by $m_i \times m_j$ matrices. By the Gauss decomposition of such a matrix we mean its representation as the product of a lower triangular block matrix with the unit matrices on the diagonal, an upper triangular block matrix with the unit matrices on the diagonal, and a block diagonal matrix, taken in some order. In this appendix we construct the Gauss decomposition of 2×2 and 3×3 matrices.

Consider first a nondegenerate 2×2 block matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (\text{A.1})$$

Suppose that the matrix a can be represented as

$$a = n_- h n_+^{-1}, \quad (\text{A.2})$$

where n_- , h and n_+ are block matrices of the form

$$n_- = \begin{pmatrix} I_{m_1} & 0 \\ n_{-12} & I_{m_2} \end{pmatrix}, \quad h = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix}, \quad n_+ = \begin{pmatrix} I_{m_1} & n_{+12} \\ 0 & I_{m_2} \end{pmatrix}.$$

Note that since the matrix a is nondegenerate, the matrices h_{11} and h_{22} must be also nondegenerate. It can be easily shown that

$$n_+^{-1} = \begin{pmatrix} I_{m_1} & -n_{+12} \\ 0 & I_{m_2} \end{pmatrix}.$$

Using this relation, one can write equality (A.2) in the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & -h_{11}n_{+12} \\ n_{21}h_{11} & h_{22} - n_{-21}h_{11}n_{+12} \end{pmatrix}.$$

From this equality it follows that the Gauss decomposition of the matrix a of form (A.1) exists if and only if the matrix a_{11} is nondegenerate. In this case one has

$$n_{-21} = a_{21}(a_{11})^{-1}, \quad (\text{A.3})$$

$$h_{11} = a_{11}, \quad (\text{A.4})$$

$$h_{22} = a_{22} - a_{21}(a_{11})^{-1}a_{12}, \quad (\text{A.5})$$

$$n_{+12} = -(a_{11})^{-1}a_{12}. \quad (\text{A.6})$$

It is worth to note that the obtained Gauss decomposition is unique.

Consider now a nondegenerate 3×3 matrix

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Suppose that a can be represented in form (A.2), where

$$h = \begin{pmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{33} \end{pmatrix},$$

$$n_- = \begin{pmatrix} I_{m_1} & 0 & 0 \\ n_{-21} & I_{m_2} & 0 \\ n_{-31} & n_{-32} & I_{m_3} \end{pmatrix}, \quad n_+ = \begin{pmatrix} I_{m_1} & n_{+12} & n_{+13} \\ 0 & I_{m_2} & n_{+23} \\ 0 & 0 & I_{m_3} \end{pmatrix}.$$

Now we have

$$n_+^{-1} = \begin{pmatrix} I_{m_1} & -n_{+12} & -n_{+13} + n_{+12}n_{+23} \\ 0 & I_{m_2} & -n_{+23} \\ 0 & 0 & I_{m_3} \end{pmatrix},$$

and in the same way as it was done for the case of 2×2 block matrices one finds out that the Gauss decomposition in question exists if and only if the matrices a_{11} and $a_{22} - a_{21}(a_{11})^{-1}a_{12}$ are nondegenerate. The explicit expressions determining the matrices n_- , h and n_+ are

$$n_{-21} = a_{21}(a_{11})^{-1}, \quad (\text{A.7})$$

$$n_{-31} = a_{31}(a_{11})^{-1}, \quad (\text{A.8})$$

$$n_{-32} = (a_{32} - a_{31}(a_{11})^{-1}a_{12})(a_{22} - a_{21}(a_{11})^{-1}a_{12})^{-1}, \quad (\text{A.9})$$

$$h_{11} = a_{11}, \quad (\text{A.10})$$

$$h_{22} = a_{22} - a_{21}(a_{11})^{-1}a_{12}, \quad (\text{A.11})$$

$$h_{33} = a_{33} - a_{31}(a_{11})^{-1}a_{13} + (a_{32} - a_{31}(a_{11})^{-1}a_{12}) \\ \times (a_{22} - a_{21}(a_{11})^{-1}a_{12})^{-1}(a_{23} - a_{21}(a_{11})^{-1}a_{13}), \quad (\text{A.12})$$

$$n_{+12} = -(a_{11})^{-1}a_{12}, \quad (\text{A.13})$$

$$n_{+13} = -(a_{11})^{-1}(a_{13} - a_{12}(a_{22} - a_{21}(a_{11})^{-1}a_{12})^{-1} \\ \times (a_{23} - a_{21}(a_{11})^{-1}a_{13})), \quad (\text{A.14})$$

$$n_{+23} = -(a_{22} - a_{21}(a_{11})^{-1}a_{12})^{-1}(a_{23} - a_{21}(a_{11})^{-1}a_{13}). \quad (\text{A.15})$$

The Gauss decomposition in this case is again unique.

It is interesting to compare the relations obtained in this appendix with those arising in the theory of quasideterminants of matrices over an associative unital ring, in the form proposed by I. M. Gelfand and V. S. Retakh [9, 10], see also [3]. Let us give here some relevant definitions.

Let \mathcal{I} and \mathcal{J} be two ordered sets, each consisting of p elements. Consider an invertible matrix $a = (a_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ with the matrix elements belonging to some associative unital ring R . Define the family of p^2 elements $|a|_{ij}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, of the ring R , which are called the quasideterminants of a . In the case $p = 1$ there is only one quasideterminant defined by $|a|_{ij} = a_{ij}$. For $p > 1$ the quasideterminant $|a|_{ij}$ of the matrix a can be defined by the relation

$$|a|_{ij}^{-1} = (a^{-1})_{ji}. \quad (\text{A.16})$$

It is clear that the quasideterminant $|a|_{ij}$ exists only if the matrix element $(a^{-1})_{ji}$ is an invertible element of the ring R . A more general definition of a quasideterminant applicable to the case of noninvertible matrices can be found in [10], but for our purposes the above definition is most convenient.

Let \mathcal{K} and \mathcal{L} be subsets of \mathcal{I} and \mathcal{J} respectively. Denote by $a^{(\mathcal{K}; \mathcal{L})}$ the matrix which is obtained from the matrix a by removing the matrix elements a_{ij} with $i \in \mathcal{K}$ or $j \in \mathcal{L}$. Consider the matrix $a^{(i; j)}$ and suppose that it is invertible. It is not difficult to show that

$$|a|_{ij} = a_{ij} - \sum_{k \neq i, l \neq j} a_{ik}(a^{(i; j)})^{-1}_{kl}a_{lj}. \quad (\text{A.17})$$

This equality is used in appendix B to prove the validity of relation (3.33).

Further, denote by $a_{(\mathcal{K}; \mathcal{L})}$ the submatrix of a composed of the matrix elements a_{ij} with $i \in \mathcal{K}$ and $j \in \mathcal{L}$. Consider the case when $\mathcal{I} = \mathcal{J} = \{1, \dots, p\}$. It can be shown that an invertible matrix $a = (a_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ has the Gauss decomposition of form (A.2) if and only if there exist the quasideterminants $|a_{(1, \dots, i; 1, \dots, i)}|_{ii}$, $i = 1, \dots, p$. Here the nontrivial matrix elements of the matrix h are determined by these quasideterminants. Actually, one has

$$h_{ii} = |a_{(1, \dots, i; 1, \dots, i)}|_{ii}. \quad (\text{A.18})$$

Returning to the case of block matrices, suppose that all blocks of the matrices under consideration are square $m \times m$ matrices. In this case we can treat such a block matrix as a matrix over the ring of $m \times m$ matrices and use the formulae relevant for matrices over the general associative unital ring.

B Proof of relation (3.33)

In this appendix we prove relation (3.33) which gives the general solution to equations (3.30)–(3.32). Probably our proof is not the shortest one but it is quite direct.

First of all note that from (3.31) it follows that

$$\beta_{i+1} = -\partial_+ \partial_- \beta_i + \partial_+ \beta_i \beta_i^{-1} \partial_- \beta_i + \beta_i \beta_{i-1}^{-1} \beta_i.$$

Therefore, to prove (3.33) it suffices to prove the equality

$$\Delta_{i+1} = \partial_+ \partial_- \Delta_i - \partial_+ \Delta_i \Delta_i^{-1} \partial_- \Delta_i + \Delta_i \Delta_{i-1}^{-1} \Delta_i. \quad (\text{B.1})$$

Let $\rho = (\rho_{rs})_{r,s=1,2,\dots}$ be the infinite block matrix with the matrix elements defined by

$$\rho_{rs} = \partial_+^{r-1} \partial_-^{s-1} \beta,$$

and let $\overset{(i)}{\rho}$ be a submatrix of ρ formed by the matrix elements ρ_{rs} with $r, s \leq i$. Recalling the definition of Δ_i one can write $\Delta_i = |\overset{(i)}{\rho}|_{ii}$. Represent the matrix $\overset{(i)}{\rho}$ in the form

$$\overset{(i)}{\rho} = \begin{pmatrix} \overset{(i-1)}{\rho} & \overset{(i-1)}{\tau} \\ \overset{(i-1)}{\sigma} & \rho_{ii} \end{pmatrix}, \quad (\text{B.2})$$

where the matrix elements of $1 \times (i-1)$ matrix $\overset{(i-1)}{\sigma}$ and $(i-1) \times 1$ matrix $\overset{(i-1)}{\tau}$ are given by

$$\left(\overset{(i-1)}{\sigma} \right)_a = \left(\overset{(i)}{\rho} \right)_{ia} = \rho_{ia}, \quad \left(\overset{(i-1)}{\tau} \right)_a = \left(\overset{(i)}{\rho} \right)_{ai} = \rho_{ai}.$$

From equality (A.17) we obtain the following expression:

$$\Delta_i = \rho_{ii} - \overset{(i-1)}{\sigma} \overset{(i-1)}{\tau} \overset{(i-1)}{\rho}^{-1}. \quad (\text{B.3})$$

Using this expression and representation (B.2) we come to the relation

$$\overset{(i)}{\rho}^{-1} = \left(\begin{array}{c|c} \overset{(i-1)}{\rho}^{-1} - \overset{(i-1)}{\rho}^{-1} \overset{(i-1)}{\tau} \overset{(i-1)}{\sigma} \overset{(i-1)}{\rho}^{-1} & -\overset{(i-1)}{\rho}^{-1} \overset{(i-1)}{\tau} \overset{(i-1)}{\Delta_i}^{-1} \\ \hline -\overset{(i-1)}{\Delta_i}^{-1} \overset{(i-1)}{\sigma} \overset{(i-1)}{\rho}^{-1} & \overset{(i-1)}{\Delta_i}^{-1} \end{array} \right). \quad (\text{B.4})$$

For the quasideterminant Δ_{i+1} one has

$$\Delta_{i+1} = \rho_{i+1,i+1} - \overset{(i)}{\sigma} \overset{(i)}{\rho}^{-1} \overset{(i)}{\tau}.$$

Note that there are valid the following relations:

$$\partial_+ \rho_{rs} = \rho_{r+1,s}, \quad \partial_- \rho_{rs} = \rho_{r,s+1}.$$

Therefore, one can represent $\overset{(i)}{\sigma}$ and $\overset{(i)}{\tau}$ as

$$\overset{(i)}{\sigma} = \begin{pmatrix} \partial_+ \overset{(i-1)}{\sigma} & \partial_+ \rho_{ii} \end{pmatrix}, \quad \overset{(i)}{\tau} = \begin{pmatrix} \partial_- \overset{(i-1)}{\tau} \\ \partial_- \rho_{ii} \end{pmatrix}.$$

Using this representation and relation (B.4) one easily obtains

$$\begin{aligned}\Delta_{i+1} &= \partial_+ \partial_- \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \\ &\quad - \left(\partial_+ \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \tau^{(i-1)} \right) \Delta_i^{-1} \left(\partial_- \rho_{ii} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \right).\end{aligned}\quad (\text{B.5})$$

Differentiating (B.3) over z^- we come to the relation

$$\begin{aligned}\partial_- \Delta_i &= \partial_- \rho_{ii} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \\ &\quad - \left(\partial_- \sigma^{(i-1)} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \rho^{(i-1)} \right) \rho^{(i-1)-1} \tau^{(i-1)}.\end{aligned}\quad (\text{B.6})$$

It is easy to get convinced that

$$\left(\partial_- \rho^{(i-1)-1} \partial_- \rho^{(i-1)} \right)_{ab} = \begin{cases} \delta_{a,b+1}, & b \neq i-1; \\ \sum_{c=1}^{i-1} \left(\partial_- \rho^{(i-1)-1} \right)_{ac} \rho_{ci}, & b = i-1. \end{cases}$$

This relation implies the equality

$$\left(\partial_- \sigma^{(i-1)} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \rho^{(i-1)} \right)_a = \Delta_i \delta_{i-1,a}.$$

Therefore, from (B.6) it follows that

$$\partial_- \Delta_i = \partial_- \rho_{ii} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} - \Delta_i \left(\rho^{(i-1)-1} \tau^{(i-1)} \right)_{i-1}. \quad (\text{B.7})$$

Now differentiate (B.3) over z^+ and use the relation

$$\left(\partial_+ \rho^{(i-1)(i-1)-1} \right)_{ab} = \begin{cases} \delta_{a+1,b}, & a \neq i-1; \\ \sum_{c=1}^{i-1} \rho_{ic} \left(\partial_+ \rho^{(i-1)-1} \right)_{cb}, & a = i-1. \end{cases} \quad (\text{B.8})$$

This gives

$$\partial_+ \Delta_i = \partial_+ \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \tau^{(i-1)} - \left(\sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \right)_{i-1} \Delta_i. \quad (\text{B.9})$$

Using relations (B.7) and (B.9) one obtains

$$\begin{aligned}\partial_+ \Delta_i \Delta_i^{-1} \partial_- \Delta_i &= \left(\partial_+ \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \tau^{(i-1)} \right) \Delta_i^{-1} \left(\partial_- \rho_{ii} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \right) \\ &\quad - \left(\partial_+ \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \tau^{(i-1)} \right) \left(\rho^{(i-1)-1} \tau^{(i-1)} \right)_{i-1} \\ &\quad - \left(\sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \right)_{i-1} \left(\partial_- \rho_{ii} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \right) \\ &\quad + \left(\sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \right)_{i-1} \Delta_i \left(\rho^{(i-1)-1} \tau^{(i-1)} \right)_{i-1}.\end{aligned}\quad (\text{B.10})$$

The differentiation of (B.7) over z^+ with account of (B.9) and (B.8) gives

$$\begin{aligned}
\partial_+ \partial_- \Delta_i &= \partial_+ \partial_- \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \\
&\quad - \left(\partial_+ \rho_{ii} - \partial_+ \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \tau^{(i-1)} \right) \left(\rho^{(i-1)-1} \tau^{(i-1)} \right)_{i-1} \\
&\quad - \left(\sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \right)_{i-1} \left(\partial_- \rho_{ii} - \sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \partial_- \tau^{(i-1)} \right) \\
&\quad + \left(\sigma^{(i-1)(i-1)-1} \rho^{(i-1)} \right)_{i-1} \Delta_i \left(\rho^{(i-1)-1} \tau^{(i-1)} \right)_{i-1} - \Delta_i \Delta_{i-1}^{-1} \Delta_i. \tag{B.11}
\end{aligned}$$

Now with (B.5), (B.11) and (B.10) we make sure that relation (B.1) is valid. Hence, relation (3.33) is also valid.

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